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H_∞ Feedback for Attitude Control of Liquid-Filled Spacecraft

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An H_∞, that is, Hardy infinity, feedback controller for the attitude control problem of a liquid-filled spacecraft is presented. The spacecraft is modeled as a main rigid body filled with an ideal liquid in uniform vortex motion and three control torques and disturbances. Quaternions are used to describe the evolution of liquid-filled spacecraft orientation to eliminate the singularities due to Euler angle representations for the kinematics motions. The H_∞ feedback controller is formulated by solving the Hamilton–Jacobi–Issacs inequality associated with the H_∞ suboptimal control problem on state manifolds according to Van der Schaft's theory (Van der Schaft, A. J., "On a State Space Approach to Non-Linear H_∞ Control," *Systems and Control Letters*, Vol. 16, No. 1, 1991, pp. 1–8 and Van der Schaft, A. J., "L_2-Gain Analysis of Non-Linear Systems and Non-Linear State Feedback H_∞ Control," *IEEE Transactions on Automatic Control*, Vol. 37, No. 6, 1992, pp. 770–784). The orientation and angular velocities of the liquid-filled spacecraft are stabilized by appropriately choosing the feedback coefficients. The determination of the coefficients is given explicitly. The numerical simulations show that the designed feedback laws can be effectively applied to stabilize the attitude of liquid-filled spacecraft with energy dissipation and external disturbances.

Introduction

THE motion of liquid-filled solids began to attract the attention of scientists more than a hundred years ago. The motion is described by a set of complicated equations consisting of nonlinear ordinary differential equations for the rigid body and partial differential equations for the liquids contained in the tanks supplemented appropriately by initial and boundary conditions. Rumyantsev[1] made a systematic research about the motion of a rigid body whose cavities are filled with homogeneous liquids, assuming that the whole system had an infinite number of degrees of freedom or a finite number of degrees of freedom. Rumyantsev[1] and Stewartson[2] developed a stability criterion for a spinning top containing liquids. Pfeiffer[3] successfully studied the stability of a variety of liquid-filled satellite attitude dynamics using methods of residues and of envelope assuming that the motion of contained liquids is of uniform vortices. Agrawal[4] investigated the dynamic characteristics of a spinning spacecraft partially filled with a liquid using the finite element method. Wie[5] described a rigid body with a spherical, dissipative fuel slug. For such a simplified, semirigid spinning spacecraft, a small change in initial conditions can lead to a change in the final spin polarity for an angular velocity component. Such sensitive dependence on initial conditions often characterizes the occurrence of chaotic dynamics. Livnel and Wie[6] investigated the spinning motion of a generic semirigid body with energy dissipation and constant body-fixed torque about one of its principal axes. Their results based on extensive simulation of a general semirigid body with constant torques about either its major, intermediate, or minor axis showed also that the polarity of the final equilibrium point is sensitive to the initial conditions.

It is well known that a spacecraft spinning about its minor axis in the presence of dissipative energy will eventually be reoriented to spin about its major axis. One practical example is the U.S. Explorer I launched in 1958, which tumbled after only a few hours of flight. The dissipating energy resulting from the four flexible antennas caused a transfer of body spin axis from the axis of minimum inertia to a transverse axis of maximum inertia. Nowadays the dynamics of control of the spacecraft have become a hot issue in spacecraft engineering.[7–10] Various control algorithms have been proposed based on either the linearized model or the feedback linearization model using optimal control techniques,[10] the variable-structure sliding techniques,[11] or the standard H_∞ optimal control strategies.[12] For reducing the effects of solar array oscillations on the Hubble Space Telescope pointing jitters, Wie and Liu[12] employed H_∞ control design methodologies to deal with its control redesign problem successfully with a noncollocated actuator and a sensor pair by using a linearization model for the pitch dynamics.

Instead of linearization, Dalsmo and Egeland[13] and Kang[14] studied a particular nonlinear H_∞ suboptimal control problem associated with a solution of the nonlinear Hamilton–Jacobi–Issacs inequalities. Dalsmo and Egeland[13] obtained the global solution from the fundamental idea of Kang,[14] who obtained the local solution of a state feedback H_∞ suboptimal control problem with respect to the rotational rigid spacecraft. Note that the H_∞ suboptimal control law by solving the associated Hamilton–Jacobi–Issacs inequalities has the same mathematical expression as formulated by Mortenson[15] using a direct method and by Vadali[11] using the equivalent variable-structural control theory. The nonlinear H_∞ control problem is mathematically based on the solutions of the Hamilton–Jacobi–Issacs inequalities, which are nonlinear partial differential inequalities. It is very difficult to determine these general inequalities. Van der Schaft[16,17] unified the results on a L_2-gain analysis of the smooth nonlinear systems and extended the relevant previous results by using an approach based on the Hamilton–Jacobi equation and inequalities and their relation to the invariant manifolds of an associated Hamiltonian vector field. He established a nonlinear analogy of the simplest part of the developed state-space approach to the linear state feedback H_∞ optimal control problem. Isidori[18] and Isidori and Astolfi[19] gave a solution to the problem of disturbance attenuation via measurement feedback with internal stability for an affine nonlinear system. They interpreted the concept of disturbance attenuation in the sense of truncated L_2 norms in the nonlinear setup.

Because of the nonlinearities associated with the differential equations of the attitude motions of the rotational spacecraft, the control problem of the spacecraft is very much involved in the complex dynamics of the nonlinear dynamical system. In 1994, Hall and Rand[20] researched the spinup nonlinear dynamics of axial dual-spin spacecraft. In 1998, Or[21] investigated the chaotic motions of a dual-spinner subject to the action of an internal oscillatory torque and the coulomb friction between the two linked bodies by employing Melnikov's method. Recently, Hammett et al.[22] established rigorously the connections between the controllability of the state-dependent factorizations and the true system controllability and introduced a notion of nonlinear stabilizability that is a necessary condition for the global closed-loop stability. Their theory is illustrated by an application to a five-state nonlinear model of a dual-spin

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spacecraft. Their research introduced the attitude nonlinear dynamics studies of the liquid-filled spacecraft.

In this paper, the model of a simpler liquid-filled spacecraft with three control torques and external disturbance inputs is considered. The paper is organized as follows. A spacecraft model with uniform vortex motion of a liquid in an ellipsoidal cavity is reviewed, and the main goal of the design is presented. Some results about state feedback H_∞ suboptimal control of an affine nonlinear system are recalled. A state feedback H_∞ suboptimal control problem for the liquid-filled spacecraft attitude is addressed, and the main results on how to choose the feedback control coefficients are given. Finally, a simple example is given to show the effectiveness when the designed control laws are applied to the attitude stabilization of the liquid-filled spacecraft with energy dissipation and external disturbances.

Equations of Rotational Motions of a Liquid-Filled Spacecraft

To simplify the dynamics equations of a liquid-filled spacecraft in this section, it is assumed that the spacecraft is completely filled with an ideal liquid in uniform vortex motion in an ellipsoidal cavity. Let $O\xi\eta\zeta$ be a space-fixed system of coordinates; $Oxyz$ be a body-fixed system of coordinates coinciding with principal axes of the ellipsoidal cavity; a_x, a_y, and a_z be the three semiaxis lengths of the ellipsoid; O be the center of mass of the system; $(\omega_x, \omega_y, \omega_z)$ be an angular velocity vector of the main body; $(\Omega_x, \Omega_y, \Omega_z)$ be a Helmholtz uniform vortex vector of a liquid contained in the ellipsoidal cavity; I_x, I_y, and I_z be the sums of moments of inertia of the centered rigid body and Zhukovsky equivalent solid body of the liquid filled in the ellipsoidal cavity; and J_x, J_y, and J_z be the difference between the moments of inertia of a consolidated liquid and of Zhukovsky equivalent solid body of the liquid contained in the ellipsoidal cavity (see Fig. 1). The problem of the angular motion of the liquid-filled spacecraft is investigated.

Based on the assumption of motion of an ideal liquid in the ellipsoidal cavity in uniform vortex motion, the motion of the ideal liquid is fully determined by a finite number of variables Ω_x, Ω_y, and Ω_z. The Helmholtz equations with respect to the variables Ω_x, Ω_y, and Ω_z are as follows[1]:

$$\frac{\mathrm{d}\Omega_x}{\mathrm{d}t} = 2a_x^2(A_{xy}\omega_z\Omega_y - A_{xz}\omega_y\Omega_z) - 2\Omega_y\Omega_z a_x^2\left(a_z^2 - a_y^2\right)A_{xy}A_{xz} \tag{1}$$

$$\frac{\mathrm{d}\Omega_y}{\mathrm{d}t} = 2a_y^2(A_{yz}\omega_x\Omega_z - A_{xy}\omega_z\Omega_x) - 2\Omega_z\Omega_x a_y^2\left(a_x^2 - a_z^2\right)A_{yz}A_{xy} \tag{2}$$

$$\frac{\mathrm{d}\Omega_z}{\mathrm{d}t} = 2a_z^2(A_{xz}\omega_y\Omega_x - A_{yz}\omega_x\Omega_y) - 2\Omega_x\Omega_y a_z^2\left(a_y^2 - a_x^2\right)A_{xz}A_{yz} \tag{3}$$

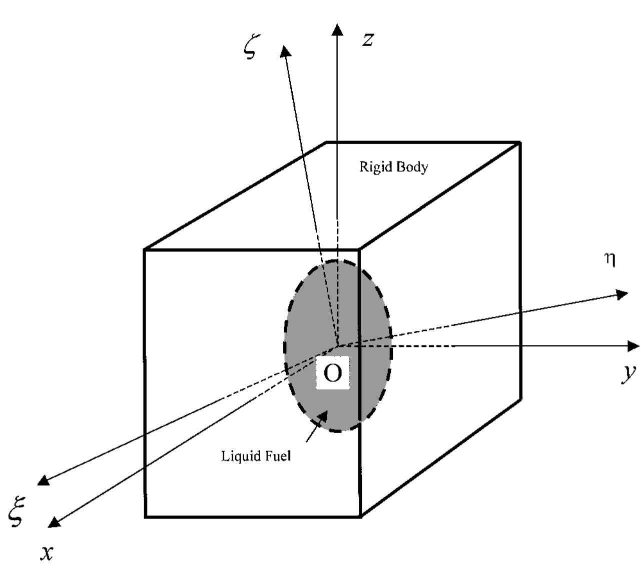

Fig. 1 Configuration of the liquid-filled spacecraft.

where

$$A_{yz} = 1/\left(a_y^2 + a_z^2\right), \qquad A_{xz} = 1/\left(a_x^2 + a_z^2\right)$$

$$A_{xy} = 1/\left(a_x^2 + a_y^2\right)$$

From the theorem of moment of momentum of a liquid-filled system, the equations of motion about the fixed point O of the liquid-filled spacecraft can be described as

$$\begin{aligned}\frac{\mathrm{d}\omega_x}{\mathrm{d}t} &= \frac{1}{I_x}\Big[u_x + d_x + (I_y - I_z)\omega_y\omega_z \\ &\quad + 2J_x a_x^2 A_{xy}A_{xz}\left(a_z^2 - a_y^2\right)\Omega_y\Omega_z + \left(2J_x a_x^2 A_{xz} - J_z\right)\omega_y\Omega_z \\ &\quad + \left(-2J_x a_x^2 A_{xy} + J_y\right)\omega_z\Omega_y\Big]\end{aligned} \tag{4}$$

$$\begin{aligned}\frac{\mathrm{d}\omega_y}{\mathrm{d}t} &= \frac{1}{I_y}\Big[u_y + d_y + (I_z - I_x)\omega_z\omega_x \\ &\quad + 2J_y a_y^2 A_{yz}A_{xy}\left(a_x^2 - a_z^2\right)\Omega_x\Omega_z + \left(-2J_y a_y^2 A_{yz} + J_z\right)\omega_x\Omega_z \\ &\quad + \left(2J_y a_y^2 A_{xy} - J_x\right)\omega_z\Omega_x\Big]\end{aligned} \tag{5}$$

$$\begin{aligned}\frac{\mathrm{d}\omega_z}{\mathrm{d}t} &= \frac{1}{I_z}\Big[u_z + d_z + (I_x - I_y)\omega_x\omega_y \\ &\quad + 2J_z a_z^2 A_{xz}A_{yz}\left(a_y^2 - a_x^2\right)\Omega_x\Omega_y + \left(2J_z a_z^2 A_{yz} - J_y\right)\omega_x\Omega_y \\ &\quad + \left(-2J_z a_z^2 A_{xz} + J_x\right)\omega_y\Omega_x\Big]\end{aligned} \tag{6}$$

where u_x, u_y, and u_z are the external torque components; d_x, d_y, and d_z are the external disturbance torque components along the body-fixed axes, respectively; and

$$J_x = 0.8M_f a_y^2 a_z^2 A_{yz}, \qquad J_y = 0.8M_f a_x^2 a_z^2 A_{xz}$$

$$J_z = 0.8M_f a_x^2 a_y^2 A_{xy}$$

where M_f is the mass of the filled liquid in the ellipsoidal tank.

To eliminate the singularities of the kinematics equations resulting from Euler angle representations, hereinafter the evolution of liquid-filled spacecraft orientation in terms of the quaternions β_x, β_y, β_z, and β_0 is described as follows:

$$\frac{\mathrm{d}\beta_x}{\mathrm{d}t} = \frac{1}{2}(\omega_x\beta_0 + \omega_z\beta_y - \omega_y\beta_z) \tag{7}$$

$$\frac{\mathrm{d}\beta_y}{\mathrm{d}t} = \frac{1}{2}(\omega_y\beta_0 - \omega_z\beta_x + \omega_x\beta_z) \tag{8}$$

$$\frac{\mathrm{d}\beta_z}{\mathrm{d}t} = \frac{1}{2}(\omega_z\beta_0 + \omega_y\beta_x - \omega_x\beta_y) \tag{9}$$

$$\frac{\mathrm{d}\beta_0}{\mathrm{d}t} = -\frac{1}{2}(\omega_x\beta_x + \omega_y\beta_y + \omega_z\beta_z) \tag{10}$$

where the relation of the quaternion components is expressed as

$$\beta_x^2 + \beta_y^2 + \beta_z^2 + \beta_0^2 = 1 \tag{11}$$

Under the assumption of no external disturbances, the design criterion of the attitude control is that the closed-loop system must have exactly one equilibrium point, namely, when the inertia principal axes of the liquid-filled body and the inertial coordinate system coincide. A desired equilibrium point is assumed to be

$$\omega_x = \omega_y = \omega_z = 0 \tag{12}$$

$$\Omega_x = 0, \qquad \Omega_y = 0, \qquad \Omega_z = \Omega_{ze} \tag{13}$$

$$\beta_x = \beta_y = \beta_z = 0, \qquad \beta_0 = 1 \tag{14}$$

where the constant Ω_{ze} must satisfy the following relation:

$$a_x^2 a_y^2 \Omega_{ze}^2 = a_y^2 a_z^2 \Omega_{x0}^2 + a_x^2 a_z^2 \Omega_{y0}^2 + a_x^2 a_y^2 \Omega_{z0}^2 \tag{15}$$

where Ω_{x0}, Ω_{y0}, and Ω_{z0} are the initial values of the components of the Helmholtz uniform vortex vector. In the control application referred to in this paper, the plant is defined by its 10-dimensional state ordinary differential equations. Equations (1–6) are for the three components of the uniform vortex vector of the contained liquid and the three components of the angular velocity vector of the main body. Equations (7–10) are the adjoined four-dimensional differential equations for the quaternions subjected to the constraint equation (11). In the following section we will present the methodology of the H_∞ suboptimal controller design for the attitude problem of the liquid-filled spacecraft.

Attitude H_∞ Suboptimal Control Law of the Liquid-Filled Spacecraft

The attitude direction of the liquid-filled spacecraft is expected to be aligned with the inertial frame, that is, the desired equilibrium point is described by Eqs. (12–14). In view of the attitude H_∞ suboptimal control model of the liquid-filled spacecraft subjected to disturbances, the output function h is defined as

$$h = h(\omega_x, \omega_y, \omega_z, \Omega_x, \Omega_y, \Omega_z, \beta_0) = \sqrt{\rho_1 T(\omega_x, \omega_y, \omega_z, \Omega_x, \Omega_y, \Omega_z) + \rho_2 p^2} \tag{16}$$

where the constants $\rho_1 > 0$ and $\rho_2 > 0$, $T = T(\omega_x, \omega_y, \omega_z, \Omega_x, \Omega_y, \Omega_z)$ is defined as a positive definite function representing the level of the kinetic energy,

$$T(\omega_x, \omega_y, \omega_z, \Omega_x, \Omega_y, \Omega_z) = \tfrac{1}{2}\left[I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 + J_x\Omega_x^2 + J_y\Omega_y^2 + J_z(\Omega_z - \Omega_{ze})^2\right] \tag{17}$$

and

$$p = p(\beta_0) = 2\arccos(|\beta_0|) \le \pi\sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2} \tag{18}$$

where p is a kind of the geodesic metric[13] measuring the distance between the moving frame and the fixed frame.

The small value of the output function h implies that the liquid-filled spacecraft is rotating near the ideal attitude equilibrium equations (12–14). For the convenience of mathematical expressions we define

$$\boldsymbol{x} = [\omega_x, \omega_y, \omega_z, \Omega_x, \Omega_y, \Omega_z, \beta_x, \beta_y, \beta_z, \beta_0]^T \tag{19}$$

$$\boldsymbol{g} = \mathrm{diag}\left[(I_x)^{-1}, (I_y)^{-1}, (I_z)^{-1}, 0, 0, 0, 0, 0, 0, 0\right] \tag{20}$$

$$\boldsymbol{D} = \mathrm{diag}\left[(I_x)^{-1}, (I_y)^{-1}, (I_z)^{-1}, 0, 0, 0, 0, 0, 0, 0\right] \tag{21}$$

$$\boldsymbol{d} = [d_x, d_y, d_z, 0, 0, 0, 0, 0, 0, 0]^T \tag{22}$$

$$\boldsymbol{f}(\boldsymbol{x}) = [f_{\Omega x}, f_{\Omega y}, f_{\Omega z}, f_{\omega x}, f_{\omega y}, f_{\omega z}, f_{\beta x}, f_{\beta y}, f_{\beta z}, f_{\beta 0}]^T \tag{23}$$

where entries of the vector $\boldsymbol{f(x)}$ defined from Eqs. (1–10) are as follows:

$$f_{\Omega x} = 2a_x^2(A_{xy}\omega_z\Omega_y - A_{xz}\omega_y\Omega_z) - 2\Omega_y\Omega_z a_x^2\left(a_z^2 - a_y^2\right)A_{xy}A_{xz} \tag{24}$$

$$f_{\Omega y} = 2a_y^2(A_{yz}\omega_x\Omega_z - A_{xy}\omega_z\Omega_x) - 2\Omega_z\Omega_x a_y^2\left(a_x^2 - a_z^2\right)A_{yz}A_{xy} \tag{25}$$

$$f_{\Omega z} = 2a_z^2(A_{xz}\omega_y\Omega_x - A_{yz}\omega_x\Omega_y) - 2\Omega_x\Omega_y a_z^2\left(a_y^2 - a_x^2\right)A_{xz}A_{yz} \tag{26}$$

$$f_{\omega x} = (1/I_x)\left[(I_y - I_z)\omega_y\omega_z + 2J_x a_x^2 A_{xy}A_{xz}\left(a_z^2 - a_y^2\right)\Omega_y\Omega_z + \left(2J_x a_x^2 A_{xz} - J_z\right)\omega_y\Omega_z + \left(-2J_x a_x^2 A_{xy} + J_y\right)\omega_z\Omega_y\right] \tag{27}$$

$$f_{\omega y} = (1/I_y)\left[(I_z - I_x)\omega_z\omega_x + 2J_y a_y^2 A_{yz}A_{xy}\left(a_x^2 - a_z^2\right)\Omega_x\Omega_z + \left(-2J_y a_y^2 A_{yz} + J_z\right)\omega_x\Omega_z + \left(2J_y a_y^2 A_{xy} - J_x\right)\omega_z\Omega_x\right] \tag{28}$$

$$f_{\omega z} = (1/I_z)\left[(I_x - I_y)\omega_x\omega_y + 2J_z a_z^2 A_{xz}A_{yz}\left(a_y^2 - a_x^2\right)\Omega_x\Omega_y + \left(2J_z a_z^2 A_{yz} - J_y\right)\omega_x\Omega_y + \left(-2J_z a_z^2 A_{xz} + J_x\right)\omega_y\Omega_x\right] \tag{29}$$

$$f_{\beta x} = \tfrac{1}{2}(\omega_x\beta_0 + \omega_z\beta_y - \omega_y\beta_z) \tag{30}$$

$$f_{\beta y} = \tfrac{1}{2}(\omega_y\beta_0 - \omega_z\beta_x + \omega_x\beta_z) \tag{31}$$

$$f_{\beta z} = \tfrac{1}{2}(\omega_z\beta_0 + \omega_y\beta_x - \omega_x\beta_y) \tag{32}$$

$$f_{\beta 0} = -\tfrac{1}{2}(\omega_x\beta_x + \omega_y\beta_y + \omega_z\beta_z) \tag{33}$$

With these definitions, the dynamical equations (1–10) of the liquid-filled spacecraft may be expressed as follows:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{gu} + \boldsymbol{Dd} \tag{34}$$

where $\boldsymbol{u}$ is the control vector defined as

$$\boldsymbol{u} = [u_x, u_y, u_z, 0, 0, 0, 0, 0, 0, 0]^T \tag{35}$$

According to the theorem established by Van der Schaft,[16,17] let constant $\gamma > 0$ and suppose that there exists a smooth solution $V(\boldsymbol{x}) \ge 0$ to the Hamiltonian–Jacobi–Issacs inequality:

$$H\left(\boldsymbol{x}, \frac{\partial V}{\partial \boldsymbol{x}}\right) = \frac{\partial V}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}) + \frac{1}{2}h^2 + \frac{1}{2}\frac{\partial V}{\partial \boldsymbol{x}}\left(\frac{1}{\gamma^2}\boldsymbol{DD}^T - \boldsymbol{gg}^T\right)\left(\frac{\partial V}{\partial \boldsymbol{x}}\right)^T \le 0 \tag{36}$$

then the closed-loop system for the feedback

$$\boldsymbol{u} = -\boldsymbol{g}^T\left(\frac{\partial V}{\partial \boldsymbol{x}}\right)^T \tag{37}$$

has the L_2 gain less than or equal to γ from the disturbances $\boldsymbol{d}$ to the block vector of the outputs h and the inputs $\boldsymbol{u}$.

For arbitrary initial conditions $\boldsymbol{x}(\boldsymbol{0})$ the following useful inequality may be obtained:

$$\int_0^R (\|h\|^2 + \|\boldsymbol{u}\|^2)\,\mathrm{d}t \le \gamma^2 \int_0^R \|\boldsymbol{d}\|^2\,\mathrm{d}t + 2V[\boldsymbol{x}(\boldsymbol{0})] \tag{38}$$

for all $R \ge 0$.

From inequality (38) the generalized gain, that is, the L_2 gain, from disturbances to liquid-filled spacecraft's energy, attitude quaternions, and control torques may be expressed as

$$\frac{\int_0^R (\|h\|^2 + \|\boldsymbol{u}\|^2)\,\mathrm{d}t - 2V[\boldsymbol{x}(\boldsymbol{0})]}{\int_0^R \|\boldsymbol{d}\|^2\,\mathrm{d}t} \le \gamma^2 \tag{39}$$

To solve the Hamilton–Jacobi–Issacs inequality (36), the following form of the solution $V(\boldsymbol{x})$ is suggested:

$$\begin{aligned} V(\boldsymbol{x}) &= V(\omega_x, \omega_y, \omega_z, \Omega_x, \Omega_y, \Omega_z, \beta_x, \beta_y, \beta_z, \beta_0) \\ &= \frac{1}{2}a\left[I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 + J_x\Omega_x^2 + J_y\Omega_y^2 + J_z(\Omega_z - \Omega_{ze})^2\right] + c\left[\beta_x^2 + \beta_y^2 + \beta_z^2 + (\beta_0 - 1)^2\right] \\ &\quad + b\left(I_x\omega_x\beta_x + I_y\omega_y\beta_y + I_z\omega_z\beta_z\right) \\ &\quad + b\left[J_x\Omega_x\beta_x + J_y\Omega_y\beta_y + J_z(\Omega_z - \Omega_{ze})\beta_z\right] \\ &\ge \frac{1}{2}[\boldsymbol{\omega}^T \quad \boldsymbol{\beta}^T]\begin{pmatrix} a\boldsymbol{I} & b\boldsymbol{I} \\ b\boldsymbol{I} & c\boldsymbol{E}\end{pmatrix}\begin{pmatrix}\boldsymbol{\omega} \\ \boldsymbol{\beta}\end{pmatrix} + \frac{1}{2}[\boldsymbol{\Omega}^T \quad \boldsymbol{\beta}^T]\begin{pmatrix} a\boldsymbol{J} & b\boldsymbol{J} \\ b\boldsymbol{J} & c\boldsymbol{E}\end{pmatrix}\begin{pmatrix}\boldsymbol{\Omega} \\ \boldsymbol{\beta}\end{pmatrix} \end{aligned} \tag{40}$$

where a, b, and c are nonnegative constants and the following notations are designated:

$$\boldsymbol{E} = \mathrm{diag}(1, 1, 1), \qquad \boldsymbol{I} = \mathrm{diag}(I_x, I_y, I_z) \tag{41}$$

$$\boldsymbol{J} = \mathrm{diag}(J_x, J_y, J_z), \qquad \boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)^T \tag{42}$$

$$\boldsymbol{\Omega} = [\Omega_x, \Omega_y, (\Omega_z - \Omega_{ze})]^T, \qquad \boldsymbol{\beta} = (\beta_x, \beta_y, \beta_z)^T \tag{43}$$

By the use of the standard matrix analysis, the following sufficient conditions for $V(\boldsymbol{x})$ being positive are given as

$$a > 0, \qquad c > 0, \qquad ac\boldsymbol{E} > b^2\boldsymbol{I}, \qquad ac\boldsymbol{E} > b^2\boldsymbol{J} \tag{44}$$

By the use of straightforward calculations, the following expressions may be derived:

$$(1/\gamma^2)\boldsymbol{D}\boldsymbol{D}^T - \boldsymbol{g}\boldsymbol{g}^T = (1/\gamma^2 - 1)\mathrm{diag}\left[1/I_x^2, 1/I_y^2, 1/I_z^2, 0, 0, 0, 0, 0, 0, 0\right] \tag{45}$$

$$\frac{\partial V}{\partial \boldsymbol{x}} = \left(\frac{\partial V}{\partial \omega_x}, \frac{\partial V}{\partial \omega_y}, \frac{\partial V}{\partial \omega_z}, \frac{\partial V}{\partial \Omega_x}, \frac{\partial V}{\partial \Omega_y}, \frac{\partial V}{\partial \Omega_z}, \frac{\partial V}{\partial \beta_x}, \frac{\partial V}{\partial \beta_y}, \frac{\partial V}{\partial \beta_z}, \frac{\partial V}{\partial \beta_0}\right) \tag{46}$$

$$\frac{\partial V}{\partial \omega_x} = I_x(a\omega_x + b\beta_x) \tag{47}$$

$$\frac{\partial V}{\partial \omega_y} = I_y(a\omega_y + b\beta_y) \tag{48}$$

$$\frac{\partial V}{\partial \omega_z} = I_z(a\omega_z + b\beta_z) \tag{49}$$

$$\frac{\partial V}{\partial \Omega_x} = J_x(a\Omega_x + b\beta_x) \tag{50}$$

$$\frac{\partial V}{\partial \Omega_y} = J_y(a\Omega_y + b\beta_y) \tag{51}$$

$$\frac{\partial V}{\partial \Omega_z} = J_z[a(\Omega_z - \Omega_{ze}) + b\beta_z] \tag{52}$$

$$\frac{\partial V}{\partial \beta_x} = bI_x\omega_x + bJ_x\Omega_x \tag{53}$$

$$\frac{\partial V}{\partial \beta_y} = bI_y\omega_y + bJ_y\Omega_y \tag{54}$$

$$\frac{\partial V}{\partial \beta_z} = bI_z\omega_z + bJ_z(\Omega_z - \Omega_{ze}) \tag{55}$$

$$\frac{\partial V}{\partial \beta_0} = -2c \tag{56}$$

Therefore,

$$\begin{aligned} H\left(x, \frac{\partial V}{\partial x}\right) &= \frac{\partial V}{\partial \omega_x} f_{\omega x} + \frac{\partial V}{\partial \omega_y} f_{\omega y} + \frac{\partial V}{\partial \omega_z} f_{\omega z} + \frac{\partial V}{\partial \Omega_x} f_{\Omega x} \\ &\quad + \frac{\partial V}{\partial \Omega_y} f_{\Omega y} + \frac{\partial V}{\partial \Omega_z} f_{\Omega z} + \frac{\partial V}{\partial \beta_x} f_{\beta x} + \frac{\partial V}{\partial \beta_y} f_{\beta y} + \frac{\partial V}{\partial \beta_z} f_{\beta z} \\ &\quad + \frac{\partial V}{\partial \beta_0} f_{\beta 0} + \frac{1}{2}\left(\frac{1}{\gamma^2} - 1\right)\left[(I_x)^{-2}\left(\frac{\partial V}{\partial \omega_x}\right)^2 \right. \\ &\quad \left. + (I_y)^{-2}\left(\frac{\partial V}{\partial \omega_y}\right)^2 + (I_z)^{-2}\left(\frac{\partial V}{\partial \omega_z}\right)^2\right] + \frac{1}{2}h^2 \end{aligned} \tag{57}$$

Also for the sake of simplicity, the following notations are designated:

$$\boldsymbol{S}(\boldsymbol{\beta}) = \begin{bmatrix} 0 & -\beta_z & \beta_y \\ \beta_z & 0 & -\beta_x \\ -\beta_y & \beta_x & 0 \end{bmatrix}, \qquad \boldsymbol{S}(\boldsymbol{\omega}) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{58}$$

$$\|\boldsymbol{\omega}\|^2 = \omega_x^2 + \omega_y^2 + \omega_z^2, \qquad \|\boldsymbol{\Omega}\|^2 = \Omega_x^2 + \Omega_y^2 + (\Omega_z - \Omega_{ze})^2 \tag{59}$$

Through some calculations from Eqs. (45–56), one has

$$\begin{aligned} &\frac{\partial V}{\partial \omega_x} f_{\omega x} + \frac{\partial V}{\partial \omega_y} f_{\omega y} + \frac{\partial V}{\partial \omega_z} f_{\omega z} + \frac{\partial V}{\partial \Omega_x} f_{\Omega x} + \frac{\partial V}{\partial \Omega_y} f_{\Omega y} + \frac{\partial V}{\partial \Omega_z} f_{\Omega z} \\ &\quad = b\beta_x[\omega_y\omega_z(I_y - I_z) + J_y\omega_z\Omega_y - J_z\omega_y\Omega_z] \\ &\quad + b\beta_y[\omega_x\omega_z(I_z - I_x) + J_z\omega_x\Omega_z - J_x\omega_z\Omega_x] \\ &\quad + b\beta_z[\omega_x\omega_y(I_x - I_y) + J_x\omega_y\Omega_x - J_y\omega_x\Omega_y] \\ &\quad - 2aa_z^2J_z\Omega_{ze}\left[A_{xz}\Omega_x\omega_y - A_{yz}\omega_x\Omega_y - \left(a_y^2 - a_x^2\right)A_{xz}A_{yz}\Omega_x\Omega_y\right] \\ &\quad = -b\boldsymbol{\beta}^T\boldsymbol{S}(\boldsymbol{\omega})(\boldsymbol{I}\boldsymbol{\omega} + \boldsymbol{J}\boldsymbol{\Omega}) - bJ_z\Omega_{ze}(\beta_x\omega_y - \beta_y\omega_x) \\ &\quad - 2aa_z^2J_z\Omega_{ze}\left[A_{xz}\Omega_x\omega_y - A_{yz}\omega_x\Omega_y - \left(a_y^2 - a_x^2\right)A_{xz}A_{yz}\Omega_x\Omega_y\right] \end{aligned} \tag{60}$$

$$\begin{aligned} &\frac{\partial V}{\partial \beta_x} f_{\beta x} + \frac{\partial V}{\partial \beta_y} f_{\beta y} + \frac{\partial V}{\partial \beta_z} f_{\beta z} + \frac{\partial V}{\partial \beta_0} f_{\beta 0} \\ &\quad = \frac{1}{2}(I_x\omega_x + J_x\Omega_x)(\omega_x\beta_0 + \omega_z\beta_y - \omega_y\beta_z) \\ &\quad + \frac{1}{2}(I_y\omega_y + J_y\Omega_y)(\omega_y\beta_0 - \omega_z\beta_x + \omega_x\beta_z) \\ &\quad + \frac{1}{2}(I_z\omega_z + J_z\Omega_z)(\omega_z\beta_0 + \omega_y\beta_x - \omega_x\beta_y) \\ &\quad + c(\omega_x\beta_x + \omega_y\beta_y + \omega_z\beta_z) \\ &\quad = \frac{1}{2}b(\boldsymbol{\omega}^T\boldsymbol{I} + \boldsymbol{\Omega}^T\boldsymbol{J})[\beta_0\boldsymbol{E} + \boldsymbol{S}(\boldsymbol{\beta})]\boldsymbol{\omega} + c\boldsymbol{\beta}^T\boldsymbol{\omega} \end{aligned} \tag{61}$$

$$\begin{aligned} &\frac{1}{2}\left(\frac{1}{\gamma^2} - 1\right)\left[(I_x)^{-2}\left(\frac{\partial V}{\partial \omega_x}\right)^2 + (I_y)^{-2}\left(\frac{\partial V}{\partial \omega_y}\right)^2 + (I_z)^{-2}\left(\frac{\partial V}{\partial \omega_z}\right)^2\right] \\ &\quad = \frac{1}{2}\left(\frac{1}{\gamma^2} - 1\right)\left[(a\omega_x + b\beta_x)^2 + (a\omega_y + b\beta_y)^2 + (a\omega_z + b\beta_z)^2\right] \\ &\quad = \frac{1}{2}\left(\frac{1}{\gamma^2} - 1\right)\left(a^2\|\boldsymbol{\omega}\|^2 + ab\boldsymbol{\beta}^T\boldsymbol{\omega} + \frac{1}{2}\|\beta\|^2\right) \end{aligned} \tag{62}$$

Therefore, by substitution of Eqs. (60–62) into Eq. (57), the following formula of the function $H[\boldsymbol{x}, \partial V/\partial \boldsymbol{x}]$ may be obtained:

$$\begin{aligned} H\left(\boldsymbol{x}, \frac{\partial V}{\partial \boldsymbol{x}}\right) &= \frac{1}{2}[b\boldsymbol{\omega}^T\boldsymbol{I} + b\boldsymbol{\Omega}^T\boldsymbol{J}][\beta_0\boldsymbol{E} + \boldsymbol{S}(\beta)]\boldsymbol{\omega} \\ &\quad - b\boldsymbol{\beta}^T\boldsymbol{S}(\boldsymbol{\omega})(\boldsymbol{I}\boldsymbol{\omega} + \boldsymbol{J}\boldsymbol{\Omega}) + \frac{1}{2}a^2\left(\frac{1}{\gamma^2} - 1\right)\|\boldsymbol{\omega}\|^2 \\ &\quad + \frac{1}{2}b^2\left(\frac{1}{\gamma^2} - 1\right)\|\boldsymbol{\beta}\|^2 + c\boldsymbol{\beta}^T\boldsymbol{\omega} + ab\left(\frac{1}{\gamma^2} - 1\right)\boldsymbol{\beta}^T\boldsymbol{\omega} + \frac{1}{2}h^2 \\ &\quad - bJ_z\Omega_{ze}\beta_x\omega_y + bJ_z\Omega_{ze}\beta_y\omega_x - 2aJ_z\Omega_{ze}a_z^2(A_{xz}\omega_y\Omega_x \\ &\quad - A_{yz}\omega_x\Omega_y) + 2aJ_z\Omega_{ze}a_z^2\left(a_y^2 - a_x^2\right)A_{xz}A_{yz}\Omega_x\Omega_y \end{aligned} \tag{63}$$

If the coefficient c is chosen to be

$$c = ab(1 - 1/\gamma^2) \tag{64}$$

then the inner product terms $\boldsymbol{\beta}^T\boldsymbol{\omega}$ are eliminated. In addition,

$$\|\boldsymbol{S}(\boldsymbol{\omega})\|_\infty = \|\boldsymbol{\omega}\|, \qquad \|\boldsymbol{\beta}\| = 1, \qquad \|\beta_0\boldsymbol{E} + \boldsymbol{S}(\boldsymbol{\beta})\|_\infty = 1 \tag{65}$$

Equation (65) may be directly obtained according to the definition of H_∞ norm. By the use of the Cauchy–Schwartz inequality, the following inequalities can be easily derived:

$$|\beta^T \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{I}\boldsymbol{\omega}| \le \|\boldsymbol{I}\|_\infty \|\boldsymbol{\omega}\|^2 \tag{66}$$

$$|\beta^T \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{J}\boldsymbol{\Omega}| \le \tfrac{1}{2}\|\boldsymbol{J}\|_\infty (\|\boldsymbol{\omega}\|^2 + \|\boldsymbol{\Omega}\|^2) \tag{67}$$

$$|\boldsymbol{\omega}^T \boldsymbol{I}[\beta_0 \boldsymbol{E} + \boldsymbol{S}(\beta)]\boldsymbol{\omega}| \le \|\boldsymbol{I}\|_\infty \|\boldsymbol{\omega}\|^2 \tag{68}$$

$$|\boldsymbol{\Omega}^T \boldsymbol{J}[\beta_0 \boldsymbol{E} + \boldsymbol{S}(\beta)]\boldsymbol{\omega}| \le \tfrac{1}{2}\|\boldsymbol{J}\|_\infty (\|\boldsymbol{\omega}\|^2 + \|\boldsymbol{\Omega}\|^2) \tag{69}$$

$$|\beta_x \omega_y| \le \tfrac{1}{2}(\|\beta\|^2 + \|\omega\|^2), \qquad |\beta_y \omega_x| \le \tfrac{1}{2}(\|\beta\|^2 + \|\omega\|^2) \tag{70}$$

$$|\Omega_x \omega_y| \le \tfrac{1}{2}(\|\boldsymbol{\Omega}\|^2 + \|\omega\|^2), \qquad |\Omega_y \omega_x| \le \tfrac{1}{2}(\|\boldsymbol{\Omega}\|^2 + \|\omega\|^2) \tag{71}$$

$$|\Omega_x \Omega_y| \le \|\boldsymbol{\Omega}\|^2 \tag{72}$$

Based on inequality (18), the following inequality can be derived:

$$h^2 = \rho_1 T + \rho_2 p^2 \le \tfrac{1}{2}\rho_1 \left(\|\boldsymbol{I}\|_\infty \|\boldsymbol{\omega}\|^2 + \|\boldsymbol{J}\|_\infty \|\boldsymbol{\Omega}\|^2\right) + \rho_2 \pi^2 \|\beta\|^2 \tag{73}$$

If one substitutes inequalities (66–73) into the function $H(\boldsymbol{x}, \partial V/\partial \boldsymbol{x})$ defined in Eq. (63), the following inequality may be formulated:

$$H\left(\boldsymbol{x}, \frac{\partial V}{\partial \boldsymbol{x}}\right) \le \delta_\beta \|\beta\|^2 + \delta_\omega \|\omega\|^2 + \delta_\Omega \|\boldsymbol{\Omega}\|^2 \tag{74}$$

where

$$\delta_\beta = -\tfrac{1}{2}b^2(1 - 1/\gamma^2) + \tfrac{1}{2}\pi^2\rho_2 + bJ_z\Omega_{ze} \tag{75}$$

$$\begin{aligned}\delta_\omega &= -\tfrac{1}{2}a^2(1 - 1/\gamma^2) + \tfrac{3}{2}b\|\boldsymbol{I}\|_\infty + \tfrac{1}{4}\rho_1\|\boldsymbol{I}\|_\infty \\ &\quad + (3b/4)\|\boldsymbol{J}\|_\infty + aJ_z\Omega_{ze}a_z^2(A_{xz} + A_{yz}) + bJ_z\Omega_{ze}\end{aligned} \tag{76}$$

$$\begin{aligned}\delta_\Omega &= \tfrac{1}{4}\rho_1\|\boldsymbol{J}\|_\infty + (3b/4)\|\boldsymbol{J}\|_\infty + aJ_z\Omega_{ze}a_z^2\big[A_{xz} + A_{yz} \\ &\quad + 2\left|\left(a_y^2 - a_x^2\right)\right| A_{xz}A_{yz}\big] + bJ_z\Omega_{ze}\end{aligned} \tag{77}$$

Now the Hamilton–Jacobi–Issacs inequality (74) will be studied in detail. From Helmholtz's equations (1–3) with respect to the variables Ω_x, Ω_y, and Ω_z, the following constant can be derived easily:

$$a_y^2 a_z^2 \Omega_x^2 + a_x^2 a_z^2 \Omega_y^2 + a_x^2 a_y^2 \Omega_z^2 = \text{const} \tag{78}$$

Therefore, it is concluded that the norm $\|\boldsymbol{\Omega}\| = \sqrt{[\Omega_x^2 + \Omega_y^2 + (\Omega_z - \Omega_{ze})^2]}$ is a finite function of time t during the itinerary of the spacecraft attitude. If one chooses the constants a and b appropriately, one can prove that the Hamilton–Jacobi–Issacs inequality (74) will hold. The determination of the constants a and b is given next.

Since $\|\beta\| \le 1$ and $\|\Omega\| = \text{const} < +\infty$, to make the Hamilton–Jacobi–Issacs inequality (74) hold, one sets

$$\delta_\omega = 0 \tag{79}$$

$$\delta_\beta + \delta_\Omega \|\boldsymbol{\Omega}\|^2 = 0 \tag{80}$$

Therefore, from these two equations, the equations with respect to a and b can be derived easily as follows:

$$fa^2 + f_1 a + f_2 b + f_3 = 0 \tag{81}$$

$$fb^2 + g_1 b + g_2 a + g_3 = 0 \tag{82}$$

or

$$\begin{aligned}&f^3a^4 + \left(2f^2f_1\right)a^3 + \left[f\left(f_1^2 + 2ff_3 - f_2g_1\right)\right]a^2 \\ &\quad + [2ff_1f_3 + f_2(-f_1g_1 + f_2g_2)]a \\ &\quad + \left[ff_3^2 + f_2(-f_3g_1 + f_2g_3)\right] = 0\end{aligned} \tag{83}$$

$$\begin{aligned}&f^3b^4 + \left(2f^2g_1\right)b^3 + \left[f\left(g_1^2 + 2fg_3 - g_2f_1\right)\right]b^2 \\ &\quad + [2fg_1g_3 + g_2(-f_1g_1 + f_2g_2)]b \\ &\quad + \left[fg_3^2 + g_2(-g_3f_1 + g_2f_3)\right] = 0\end{aligned} \tag{84}$$

where

$$f = (1 - \gamma^2)/2\gamma^2 \tag{85}$$

$$f_1 = J_z\Omega_{ze}a_z^2(A_{xz} + A_{yz}) \tag{86}$$

$$f_2 = \tfrac{3}{2}\|\boldsymbol{I}\|_\infty + \tfrac{3}{4}\|\boldsymbol{J}\|_\infty + J_z\Omega_{ze} \tag{87}$$

$$f_3 = \tfrac{1}{4}\rho_1\|\boldsymbol{I}\|_\infty \tag{88}$$

$$g_1 = J_z\Omega_{ze}\|\boldsymbol{\Omega}\|^2 + \tfrac{3}{4}\|\boldsymbol{J}\|_\infty\|\boldsymbol{\Omega}\|^2 + J_z\Omega_{ze} \tag{89}$$

$$g_2 = J_z\Omega_{ze}a_z^2\left(A_{xz} + A_{yz} + 2\left|a_y^2 - a_x^2\right|A_{xz}A_{yz}\right)\|\boldsymbol{\Omega}\|^2 \tag{90}$$

$$g_3 = \tfrac{1}{4}\rho_1\|\boldsymbol{J}\|_\infty\|\boldsymbol{\Omega}\|^2 + \tfrac{1}{2}\pi^2\rho_2 \tag{91}$$

Polynomial equations (83) and (84) with respect to a and b together with inequality (44) could be used to solve a and b.

Accordingly, from Eq. (37) the state feedback $\boldsymbol{u}$ can be calculated as follows:

$$\begin{aligned}\boldsymbol{u} &= -\boldsymbol{g}^T\left(\frac{\partial V}{\partial \boldsymbol{x}}\right)^T \\ &= -\left(\frac{1}{I_x}\frac{\partial V}{\partial \omega_x}, \frac{1}{I_y}\frac{\partial V}{\partial \omega_y}, \frac{1}{I_z}\frac{\partial V}{\partial \omega_z}, 0, 0, 0, 0, 0, 0, 0\right)^T \\ &= -[(a\omega_x + b\beta_x), (a\omega_y + b\beta_y), (a\omega_z + b\beta_z), \\ &\qquad 0, 0, 0, 0, 0, 0, 0]^T\end{aligned} \tag{92}$$

Because the desired equilibrium state in Eqs. (12–14) is assumed to be zero state, the system $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{f}(\boldsymbol{x})$ having $\boldsymbol{x}$ and $\boldsymbol{f}(\boldsymbol{x})$ defined in Eqs. (19) and (23) with the block outputs $[h, \boldsymbol{u}]^T$ is zero-state observable in the neighbourhood of the zero state. Therefore, according to the corollary established by Van der Schaft,[16,17] the closed-loop liquid-filled spacecraft dynamical system $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}\boldsymbol{u} + \boldsymbol{D}\boldsymbol{d}$ is asymptotically stable when the external disturbance $\boldsymbol{d}$ is equal to zero vector.

The control law $u_x = a\omega_x + b\beta_x$, $u_y = a\omega_y + b\beta_y$, $u_z = a\omega_z + b\beta_z$ was proposed directly by Mortensen,[15] who proved the globally asymptotically stability of the closed-loop system in the case of a rigid spacecraft without external disturbances. Dalsmo and Egeland[13] and Kang[14] derived the control law of the same form as that earlier mentioned using the H_∞ suboptimal control theory in the case of the rigid spacecraft. Vadali[11] and Sira-Ramirez and Siguerdidjane[23] studied the nonlinear stability of the closed-loop system in the case of the rigid spacecraft. Another method is given here to prove the asymptotically stability of the attitude motion of the liquid-filled spacecraft under the linear feedback control torques in Eq. (92) without external disturbances. The closed-loop dynamical system consists of ordinary differential equations (1–10) in the case of $\mathrm{d}x = \mathrm{d}y = \mathrm{d}z = 0$. The possible Lyapunov function can be constructed as

$$\begin{aligned}L(\boldsymbol{x}) &= L(\omega_x, \omega_y, \omega_z, \Omega_x, \Omega_y, \Omega_z, \beta_x, \beta_y, \beta_z, \beta_0) \\ &= \tfrac{1}{2}\left[I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 + J_x\Omega_x^2 + J_y\Omega_y^2 + J_z\Omega_z^2\right] \\ &\quad + b\left[\beta_x^2 + \beta_y^2 + \beta_z^2 + (\beta_0 - 1)^2\right]\end{aligned} \tag{93}$$

The total time derivative $\mathrm{d}L/\mathrm{d}t$ can be computed by the chain rule:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \left(I_x\omega_x\frac{\mathrm{d}\omega_x}{\mathrm{d}t}\right) + \left(I_y\omega_y\frac{\mathrm{d}\omega_y}{\mathrm{d}t}\right) + \left(I_z\omega_z\frac{\mathrm{d}\omega_z}{\mathrm{d}t}\right) + \left(J_x\Omega_x\frac{\mathrm{d}\Omega_x}{\mathrm{d}t}\right) + \left(J_y\Omega_y\frac{\mathrm{d}\Omega_y}{\mathrm{d}t}\right) + \left(J_z\Omega_z\frac{\mathrm{d}\Omega_z}{\mathrm{d}t}\right) - 2b\frac{\mathrm{d}\beta_0}{\mathrm{d}t} \tag{94}$$

If one substitutes Eqs. (1–6) and (10) into Eq. (94), then the time derivative of $L(\boldsymbol{x})$ can be changed into the following form:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -a\left(\omega_x^2 + \omega_y^2 + \omega_z^2\right) \tag{95}$$

Consequently, $\mathrm{d}L/\mathrm{d}t \leq 0$ holds for all t. It is not too difficult to prove that $\mathrm{d}L/\mathrm{d}t = 0$ holds if and only if Eqs. (12–14) hold. According to Lyapunov's stability theory, the closed-loop system approaches asymptotically the desired equilibrium point starting from any initial conditions in the case of no external disturbances.

Simulation Results

A practical model of the liquid-filled spacecraft will be modeled as a rigid body with a spheroid slug of inertia matrix $\mathrm{diag}(J_{fx}, J_{fy}, J_{fz})$. The slug is centered about the center of mass of the entire spacecraft and is surrounded by a viscous layer of viscosity μ to include the impact of the viscosity on the rotational motion of a liquid-filled spacecraft subject to three body-fixed torques and external disturbances. The rotational equations of motion of such a simple semirigid spacecraft are given as follows:

$$\frac{\mathrm{d}\Omega_x}{\mathrm{d}t} = 2a_x^2(A_{xy}\omega_z\Omega_y - A_{xz}\omega_y\Omega_z) - 2\Omega_y\Omega_z a_x^2\left(a_z^2 - a_y^2\right)A_{xy}A_{xz} + \frac{\mu}{J_{fx}}(\omega_x - \Omega_x) \tag{96}$$

$$\frac{\mathrm{d}\Omega_y}{\mathrm{d}t} = 2a_y^2(A_{yz}\omega_x\Omega_z - A_{xy}\omega_z\Omega_x) - 2\Omega_z\Omega_x a_y^2\left(a_x^2 - a_z^2\right)A_{yz}A_{xy} + \frac{\mu}{J_{fy}}(\omega_y - \Omega_y) \tag{97}$$

$$\frac{\mathrm{d}\Omega_z}{\mathrm{d}t} = 2a_z^2(A_{xz}\omega_y\Omega_x - A_{yz}\omega_x\Omega_y) - 2\Omega_x\Omega_y a_z^2\left(a_y^2 - a_x^2\right)A_{xz}A_{yz} + \frac{\mu}{J_{fz}}(\omega_z - \Omega_z) \tag{98}$$

$$\frac{\mathrm{d}\omega_x}{\mathrm{d}t} = \frac{1}{I_x}\Big[u_x + d_x + (I_y - I_z)\omega_y\omega_z + 2J_x a_x^2 A_{xy}A_{xz}\left(a_z^2 - a_y^2\right)\Omega_y\Omega_z + \left(2J_x a_x^2 A_{xz} - J_z\right)\omega_y\Omega_z + \left(-2J_x a_x^2 A_{xy} + J_y\right)\omega_z\Omega_y - \mu(\omega_x - \Omega_x)\Big] \tag{99}$$

$$\frac{\mathrm{d}\omega_y}{\mathrm{d}t} = \frac{1}{I_y}\Big[u_y + d_y + (I_z - I_x)\omega_z\omega_x + 2J_y a_y^2 A_{yz}A_{xy}\left(a_x^2 - a_z^2\right)\Omega_x\Omega_z + \left(-2J_y a_y^2 A_{yz} + J_z\right)\omega_x\Omega_z + \left(2J_y a_y^2 A_{xy} - J_x\right)\omega_z\Omega_x - \mu(\omega_y - \Omega_y)\Big] \tag{100}$$

$$\frac{\mathrm{d}\omega_z}{\mathrm{d}t} = \frac{1}{I_z}\Big[u_z + d_z + (I_x - I_y)\omega_x\omega_y + 2J_z a_z^2 A_{xz}A_{yz}\left(a_y^2 - a_x^2\right)\Omega_x\Omega_y + \left(2J_z a_z^2 A_{yz} - J_y\right)\omega_x\Omega_y + \left(-2J_z a_z^2 A_{xz} + J_x\right)\omega_y\Omega_x - \mu(\omega_z - \Omega_z)\Big] \tag{101}$$

where $u_x = a\omega_x + b\beta_x$, $u_y = a\omega_y + b\beta_y$, and $u_z = a\omega_z + b\beta_z$ and the coefficients a and b are defined according to the polynomial equations (83) and (84) together with inequality (44).

For a viscous problem without external disturbances, the quantity Ω_{ze} will be expected to be 0. Construct the Lyapunov function as Eq. (93). Consider the disturbed motions of the ordinary equations (7–10) and (96–101), then the total time derivative $\mathrm{d}L/\mathrm{d}t$ can be computed by the chain rule:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\left((\mu + a)\omega_x^2 - \mu\left(1 + \frac{J_x}{J_{fx}}\right)\omega_x\Omega_x + \mu\frac{J_x}{J_{fx}}\Omega_x^2\right) - \left((\mu + a)\omega_y^2 - \mu\left(1 + \frac{J_y}{J_{fy}}\right)\omega_y\Omega_y + \mu\frac{J_y}{J_{fy}}\Omega_y^2\right) - \left((\mu + a)\omega_z^2 - \mu\left(1 + \frac{J_z}{J_{fz}}\right)\omega_z\Omega_z + \mu\frac{J_z}{J_{fz}}\Omega_z^2\right) \tag{102}$$

The sufficient conditions for the total time derivative $\mathrm{d}L/\mathrm{d}t$ to be negative is that the following three inequalities hold simultaneously:

$$4a(J_x/J_{fx}) - \mu(1 - J_x/J_{fx})^2 > 0 \tag{103}$$

$$4a(J_y/J_{fy}) - \mu(1 - J_y/J_{fy})^2 > 0 \tag{104}$$

$$4a(J_z/J_{fz}) - \mu(1 - J_z/J_{fz})^2 > 0 \tag{105}$$

To test the performance of a liquid-filled spacecraft under the feedback designed in this paper, numerical simulations were performed with the following data.

The inertia matrix (kilogram square meter) is

$$\boldsymbol{I} = \mathrm{diag}(I_x, I_y, I_z) = \mathrm{diag}(150, 150, 180) \tag{106}$$

The three semiaxis lengths (meter) of the ellipsoidal tank are

$$a_x = 0.4, \qquad a_y = 0.5, \qquad a_z = 0.6 \tag{107}$$

The density (kilogram per cubic meter) of the filled liquid fuel is

$$\rho = 1447 \tag{108}$$

The external disturbance torques (newton meter) are assumed as follows:

$$d_x = 0.01 + 0.01\sin(2\pi 0.12t) + 0.05\sin(2\pi 0.66t) \tag{109}$$

$$d_y = 0.01 + 0.01\sin(2\pi 0.12t) + 0.05\sin(2\pi 0.66t) \tag{110}$$

$$d_z = 0.01 + 0.02\sin(2\pi 0.12t) + 0.05\sin(2\pi 0.66t) \tag{111}$$

The viscosity μ between the rigid-body tank wall and the spheroid slug is assumed to be $\mu = 15\ \mathrm{N\cdot m\cdot s}$. The constant γ is selected as $\gamma = 3$. A simulation experiment demonstrates that the variations of the weighting numbers ρ_1 and ρ_2 can change the magnitudes of the control torques. Here $\rho_1 = 1$ and $\rho_2 = 10$ are chosen. The feedback control coefficients a and b can be obtained from the solutions of the polynomial equations (83) and (84). Therefore, $a = 115.88\ \mathrm{N\cdot m\cdot s}$

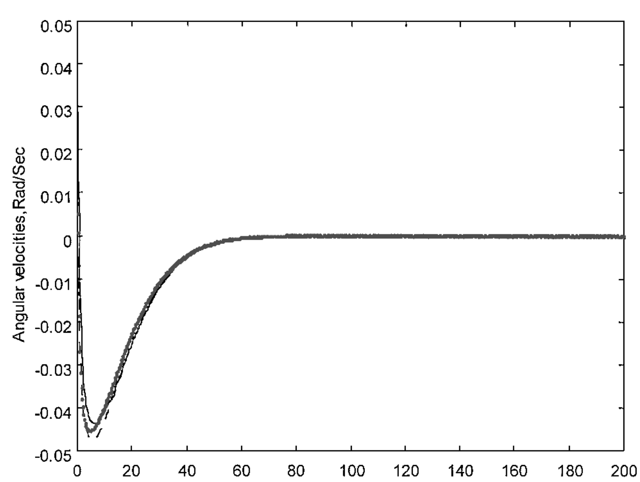

Fig. 2 Evolution of the angular velocities of the liquid-filled spacecraft: $\cdots\cdots$, ω_x; ——, ω_y; and $---$, ω_z.

and $b = 15.986$ N · m. A fourth-order Runge–Kutta integration algorithm by MATLAB¨ was used to simulate the dynamics from the nonlinear ordinary equations (96–101) and (7–10).

Figure 2 shows the evolution of the angular velocities of the liquid-filled spacecraft. Figure 3 shows the evolution of the vortices of the liquid filled in the ellipsoidal tank. Figure 4 shows the evolution of the quaternions. Figure 5 shows the control torques commanded during the control period. Figures 6–8 show the magnitudes of the commanded control torques from $t = 105$ to 200 s. Figure 9 shows the magnitudes of the angular velocities from $t = 105$ to 200 s. Figure 10 shows the magnitudes of the quaternions from $t = 105$ to 200 s. The simulation results show that, under the action of the designed torques and disturbances, the damping torques do not change the attitude quaternions and the angular velocities much. From the evolution of the angular velocities and quaternions, it is evident that the proposed controller of the liquid-filled spacecraft without external disturbances can be stabilised to the desired equilibrium. The simulations also show that inequality (38) holds during the itinerary of the control of attitude of the liquid-filled

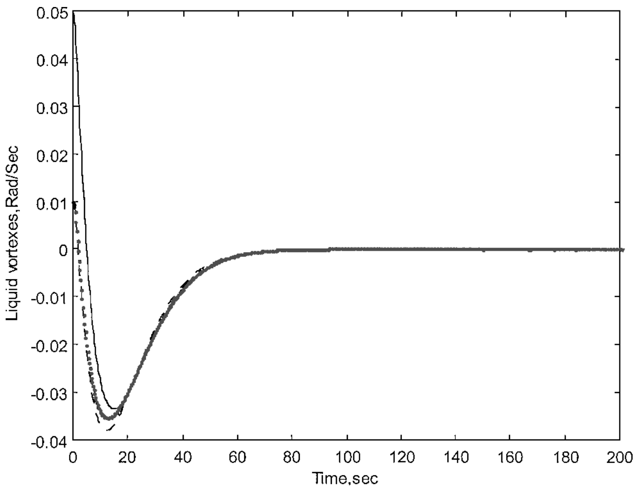

Fig. 3 Evolution of the vortices of the contained liquid fuel slug: · · · ·, Ω_x; ——, Ω_y; and – – –, Ω_z.

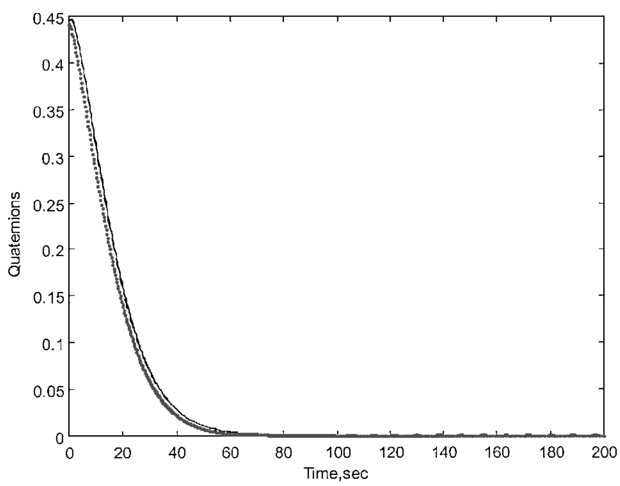

Fig. 4 Evolution of the quaternions: · · · ·, β_x; ——, β_y; and – – –, β_z.

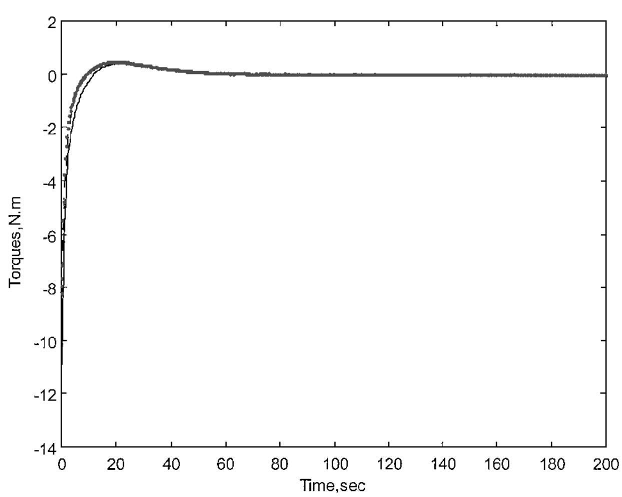

Fig. 5 Control torques commanded during the control period: · · · ·, u_x; ——, u_y; and – – –, u_z.

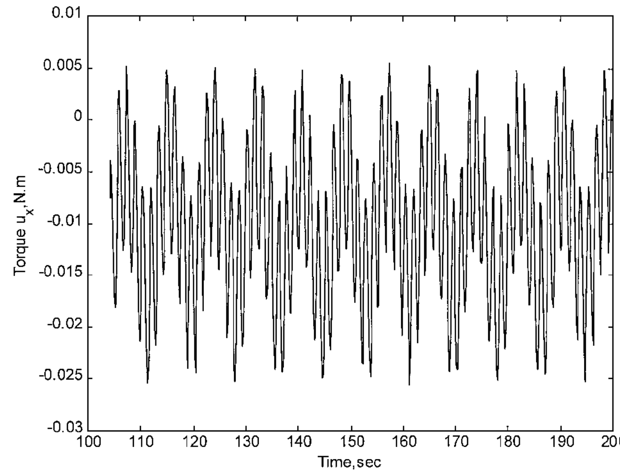

Fig. 6 Steady magnitude of commanded control torques u_x.

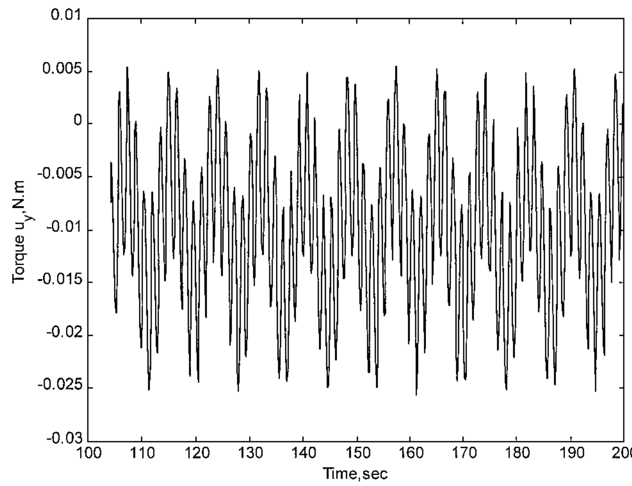

Fig. 7 Steady magnitude of commanded control torques u_y.

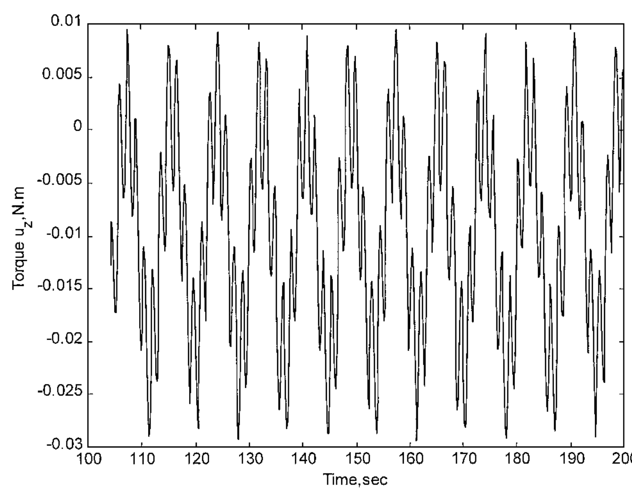

Fig. 8 Steady magnitude of commanded control torques u_z.

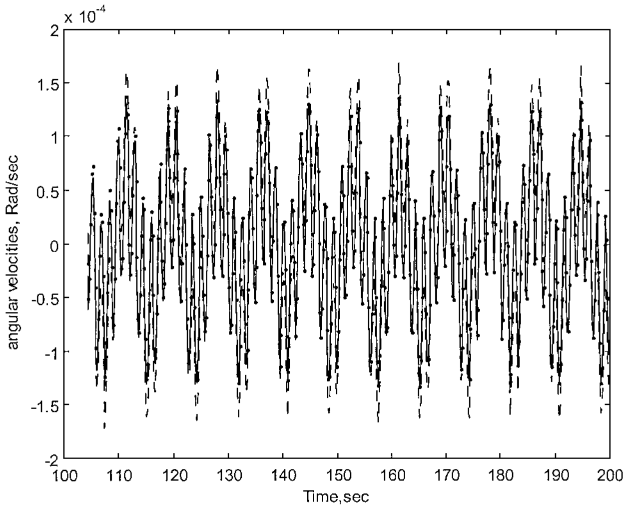

Fig. 9 Steady magnitude of angular velocities: · · · · ·, ω_x; ——, ω_y; and – – –, ω_z.

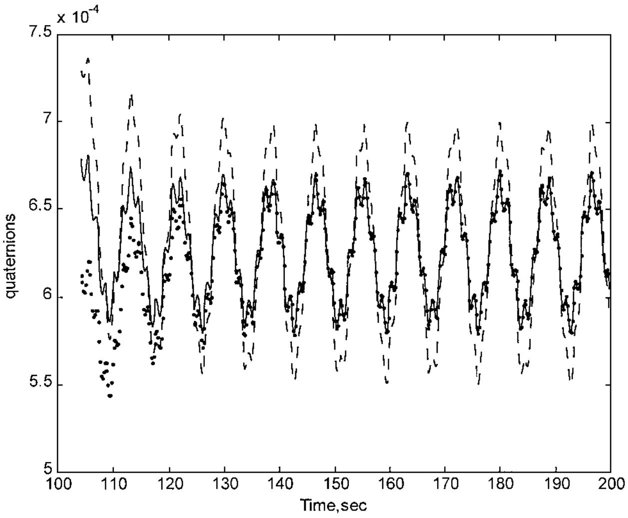

Fig. 10 Steady magnitude of quaternions: · · · · ·, β_x; ——, β_y; and – – –, β_z.

spacecraft and that after the transient time the maximum values of the control torques are around the maximum values of the disturbances.

Conclusions

The attitude feedback law for the liquid-filled spacecraft has been designed by using the theory of nonlinear Hardy-infinity optimal control theory. It is proved theoretically that the attitude quaternions and angular velocities of the liquid-filled spacecraft can be stabilized from an arbitrary initial deviating state to the desired equilibrium point. The algorithms for computing feedback coefficients are given analytically. Any number greater than one can bound the generalized gain from disturbances to liquid-filled spacecraft's energy, attitude quaternions, and control torques. Under the action of combined designed control and damping torques, the vortices of a liquid in the ellipsoidal tank are attenuated as expected. The simulation results also show that a small change of viscosity does not change much the response of the attitude quaternions and the angular velocities. After the transient response, the maximum values of the control torques are around the maximum values of the disturbances.

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